

MULTI-STATE CANALYZING FUNCTIONS OVER FINITE FIELDS

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ABSTRACT. In this paper, we extend the definition of Boolean canalizing functions to the canalizing functions over finite field \mathbb{F}_q , where q is a power of a prime. We obtain the characterization of all the eight classes of such functions as well as their cardinality. When $q = 2$, we obtain a combinatorial identity by equating our result to the formula in [2]. Finally, for a better understanding to the magnitude, we obtain the asymptotes for all the eight cardinalities as either $n \rightarrow \infty$ or $q \rightarrow \infty$.

1. INTRODUCTION

In 1993, canalizing Boolean rules were introduced by S. Kauffman [6] as biologically appropriate rules in Boolean network models of gene regulatory networks. When comparing the class of canalizing functions to other classes of functions with respect to their evolutionary plausibility as emergent control rules in genetic regulatory system, it is informative to know the number of canalizing functions with a given number of input variables [2]. However, the Boolean network modeling paradigm is rather restrictive, with its limit to two possible functional levels, ON and OFF, for genes, proteins, etc. Many discrete models of biological networks therefore allow variables to take on multiple states. Commonly used discrete multi-state model types are the so-called logical models [19], Petri nets [18], and agent-based models [17]. It was shown in [20] and [21] that many of these models can be translated into the rich and general mathematical framework of *polynomial dynamical systems over a finite field* \mathbb{F}_q . (Software to carry out this translation is available at <http://dvd.vbi.vt.edu/cgi-bin/git/adam.pl>).

In this paper, we generalize the concept of Boolean canalizing rules to the multi-state case, that is, to functions over any finite fields \mathbb{F}_q , thus generalizing the results in [2]. We provide formulas for the cardinalities of all the eight classes canalizing functions. We also obtain the asymptotes of these cardinalities as either $n \rightarrow \infty$ or $q \rightarrow \infty$.

2. PRELIMINARIES

In this section we introduce the definition of a *canalizing function*. Let $\mathbb{F} = \mathbb{F}_q$ be a finite field with q elements, where q is a power of a prime. If f is a n variable function from \mathbb{F}^n to \mathbb{F} , it is well known [11] that f can be expressed as a polynomial, called the algebraic normal form (ANF):

$$f(x_1, x_2, \dots, x_n) = \sum_{k_1=0}^{q-1} \sum_{k_2=0}^{q-1} \cdots \sum_{k_n=0}^{q-1} a_{k_1 k_2 \dots k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

where each coefficient $a_{k_1 k_2 \dots k_n} \in \mathbb{F}$ is a constant. The number $k_1 + k_2 + \cdots + k_n$ is the multivariate degree of the term $a_{k_1 k_2 \dots k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ with nonzero coefficient $a_{k_1 k_2 \dots k_n}$. The greatest degree of all the terms of f is called the algebraic degree, denoted by $\deg(f)$. The greatest degree of each individual variable x_i will be denoted by $\deg(f)_i$. Let $[n] = \{1, 2, \dots, n\}$.

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It is shown in [20] that it is no restriction of generality to consider models in which the set of states of the model variables have the algebraic structure of a finite field. The above fact that *any* function $\mathbb{F}^n \rightarrow \mathbb{F}$ can be represented as a polynomial makes the results of this paper valid in the most general setting of models that are given as dynamical systems generated by iteration of set functions.

We now define a notion of canalyzing function in multi-state setting, which is a straightforward generalization of the Boolean case.

Definition 2.1. *A function $f(x_1, x_2, \dots, x_n)$ is canalyzing in the i th variable with canalyzing input value $a \in \mathbb{F}$ and canalyzed output value $b \in \mathbb{F}$ if $f(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n) = b$, for any (x_1, x_2, \dots, x_n) .*

In other words, a function is canalyzing if there exists one variable x_i such that, if x_i receives certain inputs, this by itself determines the value of the function. For the purpose of the proofs below we will need to use families of canalyzing functions for which part of the specification is fixed, such as the variable x_i or a or b or some combination. For ease of notation, we will refer to a canalyzing function just as canalyzing if no additional information is specified. A function that is canalyzing in variable i with canalyzing input value $a \in \mathbb{F}$ and canalyzed output value $b \in \mathbb{F}$ will be referred to as $\langle i : a : b \rangle$ canalyzing.

We introduce an additional concept.

Definition 2.2. *$f(x_1, x_2, \dots, x_n)$ is essential in variable x_i if there exist $r, s \in \mathbb{F}$ such that $f(x_1, \dots, x_{i-1}, r, x_{i+1}, \dots, x_n) \neq f(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_n)$.*

Example 2.3. *let $q = 5, n = 3$. $f(x_1, x_2, x_3) = 2(x_1 - 3)^3(x_1 - 2)x_2 + 1$. Then this function is essential on x_1 and x_2 but not essential on x_3 . It has algebraic degree 5 with $\deg(f)_1 = 4$ and $\deg(f)_2 = 1$. Note that f is canalyzing in x_1 with canalyzing input value 3 and canalyzed output value 1, i.e. f is $\langle 1 : 3 : 1 \rangle$ canalyzing. Note that f is also $\langle 1 : 2 : 1 \rangle$ and $\langle 2 : 0 : 1 \rangle$ canalyzing. Since f is not essential in x_3 it cannot be $\langle 3 : a : b \rangle$ canalyzing for any $a, b \in \mathbb{F}_q$.*

If a function has exactly one essential variable, say x_i , then its ANF is

$$f = a_{q-1}x_i^{q-1} + \dots + a_1x_i + a_0$$

there exist a $j \geq 1$ such that $a_j \neq 0$. There are $q^q - q$ many such functions since all the constants should be excluded. $\frac{1}{q}(q^q - q) = q^{q-1} - 1$ many of them have fixed canalyzed value b for any $b \in \mathbb{F}$ since each number is equal. In total, there are $n(q^{q-1} - 1)$ many one essential variable canalyzing function with fixed canalyzed value b for any b since there are n variables.

There is only one constant function with fixed canalyzed value which is itself.

Notation 2.4. *For $i \in 0, 1, \dots, n$ and $a, b \in \mathbb{F}_q$ we will use the following notation,*

$\mathcal{C}_{a,b}^i$: *The set of all functions that are canalyzing in the i th variable with canalyzing input value a and canalyzed output value b .*

$\mathcal{C}_{*,b}^i$: *The set of all functions that are canalyzing in the i th variable with some canalyzing input value in \mathbb{F}_q and canalyzed output value b .*

$\mathcal{C}_{a,*}^i$: *The set of all functions that are canalyzing in the i th variable with canalyzing input value a and some canalyzed output value in \mathbb{F}_q .*

$\mathcal{C}_{a,b}^*$: *The set of all functions that are canalyzing on some variable with canalyzing input value a and canalyzed output value b .*

$\mathcal{C}_{*,*}^i$: *The set of all functions that are canalyzing in the i th variable with some canalyzing input value in \mathbb{F}_q and some canalyzed output value in \mathbb{F}_q .*

$\mathcal{C}_{a,*}^*$: *The set of all functions that are canalyzing on some variable with canalyzing input value a and some canalyzed output value in \mathbb{F}_q .*

$\mathcal{C}_{*,b}^*$: The set of all functions that are canalizing on some variable with some canalizing input value in \mathbb{F}_q and canalized output value b .

$\mathcal{C}_{*,*}^*$: The set of all functions that are canalizing on some variable with some canalizing input value in \mathbb{F}_q and some canalized output value in \mathbb{F}_q , i.e., this set consists of all the canalizing functions.

We have the following propositions.

Proposition 2.5. $\mathcal{C}_{a,b_1}^i \cap \mathcal{C}_{a,b_2}^i = \emptyset$ whenever $b_1 \neq b_2$.

Proposition 2.6. $\mathcal{C}_{*,b_1}^{i_1} \cap \mathcal{C}_{*,b_2}^{i_2} = \emptyset$ whenever $b_1 \neq b_2$ and $i_1 \neq i_2$.

Proof. Let $f \in \mathcal{C}_{*,b_1}^{i_1} \cap \mathcal{C}_{*,b_2}^{i_2}$, then there exist a_1 such that the value of f should be always b_1 if we let $x_{i_1} = a_1$. Similarly, there exist a_2 such that the value of f should be b_2 if we let $x_{i_2} = a_2$. But $b_1 \neq b_2$, a contradiction. \square

With the above notations, we have

$$\begin{aligned} \mathcal{C}_{*,*}^* &= \bigcup_{b \in \mathbb{F}} \mathcal{C}_{*,b}^* = \bigcup_{a \in \mathbb{F}} \mathcal{C}_{a,*}^* = \bigcup_{i \in [n]} \mathcal{C}_{*,*}^i, \\ \mathcal{C}_{*,b}^* &= \bigcup_{i \in [n]} \mathcal{C}_{*,b}^i = \bigcup_{a \in \mathbb{F}} \mathcal{C}_{a,b}^*, \\ \mathcal{C}_{a,*}^* &= \bigcup_{b \in \mathbb{F}} \mathcal{C}_{a,b}^* = \bigcup_{i \in [n]} \mathcal{C}_{a,*}^i, \\ \mathcal{C}_{*,*}^i &= \bigcup_{a \in \mathbb{F}} \mathcal{C}_{a,*}^i = \bigcup_{b \in \mathbb{F}} \mathcal{C}_{*,b}^i, \\ \mathcal{C}_{*,b}^i &= \bigcup_{a \in \mathbb{F}} \mathcal{C}_{a,b}^i, \\ \mathcal{C}_{a,*}^i &= \bigcup_{b \in \mathbb{F}} \mathcal{C}_{a,b}^i, \\ \mathcal{C}_{a,b}^* &= \bigcup_{i \in [n]} \mathcal{C}_{a,b}^i. \end{aligned}$$

For any set S , we use $|S|$ to stand for its cardinality. We use $C(n, k) = \frac{n!}{k!(n-k)!}$ to stand for the binomial coefficients.

Obviously, for the above notations, the cardinality are same for different values of i , a and b . In other words, we have $|\mathcal{C}_{a_1,b_1}^{i_1}| = |\mathcal{C}_{a_2,b_2}^{i_2}|$, $|\mathcal{C}_{*,b}^i| = |\mathcal{C}_{*,c}^j|$, $|\mathcal{C}_{a,b}^*| = |\mathcal{C}_{c,d}^*|$ and etc.

3. CHARACTERIZATION AND ENUMERATION OF CANALYZING FUNCTIONS OVER \mathbb{F}

Similar to [1] we have

Lemma 3.1. $f(x_1, x_2, \dots, x_n)$ is $\langle i : a : b \rangle$ canalizing iff

$$f(X) = f(x_1, x_2, \dots, x_n) = (x_i - a)Q(x_1, x_2, \dots, x_n) + b. \text{ where } \deg(Q)_i \leq q - 2.$$

Proof. From the algebraic normal form of f , we rewrite it as $f = x_i^{q-1}g_{q-1}(X_i) + x_i^{q-2}g_{q-2}(X_i) \dots + x_i g_1(X_i) + g_0(X_i)$, where $X_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Using long division we get $f(X) = f(x_1, x_2, \dots, x_n) = (x_i - a)Q(x_1, x_2, \dots, x_n) + r(X_i)$. Since $f(X)$ is $\langle i : a : b \rangle$ canalizing, we get $f(X) = f(x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_n) = b$ for any (x_1, x_2, \dots, x_n) , i.e., $r(X_i) = b$ for any X_i . So $r(X_i)$ must be the constant b . We finished the necessity. The sufficiency is obvious. \square

The above lemma means $f(X)$ is $\langle i : a : b \rangle$ iff $(x_i - a)|(f(X) - b)$.

Now we get our first formula.

Lemma 3.2. For any $i \in [n]$, $a, b \in \mathbb{F}$, there are $q^{q^n - q^{n-1}}$ many $\langle i : a : b \rangle$ canalizing functions. In other words, $|\mathcal{C}_{a,b}^i| = q^{q^n - q^{n-1}}$.

Proof. In Lemma 3.1, Q can be any polynomial with $\deg(Q)_i \leq q - 2$. Its ANF is

$\sum_{k_1=0}^{q-1} \dots \sum_{k_{i-1}=0}^{q-1} \sum_{k_{i+1}=0}^{q-1} \dots \sum_{k_n=0}^{q-1} a_{k_1 k_2 \dots k_n} x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$. Since each coefficient has q many choices and there are $(q-1)q^{n-1} = q^n - q^{n-1}$ monomials, we get what we want. \square

Because $\mathcal{C}_{a,*}^i = \bigcup_{b \in \mathbb{F}} \mathcal{C}_{a,b}^i$, by Proposition 2.5, we get

Corollary 3.3. $|\mathcal{C}_{a,*}^i| = q(q^{q^n - q^{n-1}}) = q^{q^n - q^{n-1} + 1}$.

Lemma 3.4. For any $\{a_1, a_2, \dots, a_k\} \subset \mathbb{F}$, $f(X) \in \bigcap_{j=1}^k \mathcal{C}_{a_j,b}^i$ iff

$$f(X) = f(x_1, x_2, \dots, x_n) = (\prod_{j=1}^k (x_i - a_j))Q(x_1, x_2, \dots, x_n) + b, \text{ where } \deg(Q)_i \leq q - k - 1.$$

Proof. From Lemma 3.1, we know $(x_i - a_j)|(f(X) - b)$ for $j = 1, 2, \dots, k$. So is their product since they are pairwise coprime. \square

Lemma 3.5. $|\bigcap_{j=1}^k \mathcal{C}_{a_j,b}^i| = q^{q^n - kq^{n-1}}$ for any $\{a_1, a_2, \dots, a_k\} \subset \mathbb{F}$.

Proof. This is similar to the proof of Lemma 3.2 by Lemma 3.4. \square

Note: If $k = q$, the above number is 1. This is because it means $(x_i - a_j)|(f(X) - b)$ for all the a_j , $j = 1, 2, \dots, q$, i.e., $(x_i^q - x_i)|(f(X) - b)$, where $x_i^q - x_i = \prod_{a \in \mathbb{F}} (x_i - a)$. So $f(X) - b = 0$ which means $f(x) = b$.

Theorem 3.6. For any $i \in [n]$, $b \in \mathbb{F}$, $|\mathcal{C}_{*,b}^i| = |\bigcup_{a \in \mathbb{F}} \mathcal{C}_{a,b}^i| = q^{q^n} - (q^{q^{n-1}} - 1)^q$.

Proof. By Inclusion and Exclusion Principle, we have $|\mathcal{C}_{*,b}^i| = |\bigcup_{a \in \mathbb{F}} \mathcal{C}_{a,b}^i| = \sum_{a \in \mathbb{F}} |\mathcal{C}_{a,b}^i| - \sum_{\{a_1, a_2\} \subset \mathbb{F}} |\mathcal{C}_{a_1,b}^i \cap \mathcal{C}_{a_2,b}^i| + \dots + (-1)^{k-1} \sum_{\{a_1, a_2, \dots, a_k\} \subset \mathbb{F}} |\bigcap_{j=1}^k \mathcal{C}_{a_j,b}^i| + \dots + (-1)^{q-1} = C(q, 1)q^{q^n - q^{n-1}} - C(q, 2)q^{q^n - 2q^{n-1}} + (-1)^k C(q, k)q^{q^n - kq^{n-1}} + \dots + 1 = \sum_{k=1}^q (-1)^{k-1} C(q, k)q^{q^n - kq^{n-1}} = q^{q^n} \sum_{k=1}^q (-C(q, k)(-q^{-q^{n-1}})^k) = q^{q^n} (1 - (1 - q^{-q^{n-1}})^q) = q^{q^n} - (q^{q^{n-1}} - 1)^q. \quad \square$

Similarly,

Lemma 3.7. For any $\{i_1, i_2, \dots, i_k\} \subset [n]$, $f(X) \in \bigcap_{j=1}^k \mathcal{C}_{a,b}^{i_j}$ iff

$$f(X) = f(x_1, x_2, \dots, x_n) = (\prod_{j=1}^k (x_{i_j} - a))Q(x_1, x_2, \dots, x_n) + b, \text{ where } \deg(Q)_{i_j} \leq q - 1, j = 1, 2, \dots, k.$$

Lemma 3.8. $|\bigcap_{j=1}^k \mathcal{C}_{a,b}^{i_j}| = q^{(q-1)^k q^{n-k}}$ for any $\{i_1, i_2, \dots, i_k\} \subset [n]$.

Theorem 3.9. $|\mathcal{C}_{a,b}^*| = \sum_{1 \leq k \leq n} (-1)^{k-1} C(n, k)q^{(q-1)^k q^{n-k}}$.

Proof. $|\mathcal{C}_{a,b}^*| = |\bigcup_{i \in [n]} \mathcal{C}_{a,b}^i| = \sum_{1 \leq i \leq n} |\mathcal{C}_{a,b}^i| - \sum_{1 \leq i < j \leq n} |\mathcal{C}_{a,b}^i \cap \mathcal{C}_{a,b}^j| + \dots + (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |\bigcap_{j=1}^k \mathcal{C}_{a,b}^{i_j}| + \dots + (-1)^{n-1} |\bigcap_{j=1}^n \mathcal{C}_{a,b}^j| = C(n, 1)q^{(q-1)q^{n-1}} - C(n, 2)q^{(q-1)^2 q^{n-2}} + \dots + (-1)^{k-1} C(n, k)q^{(q-1)^k q^{n-k}} + \dots + (-1)^{n-1} q^{(q-1)^n} = \sum_{1 \leq k \leq n} (-1)^{k-1} C(n, k)q^{(q-1)^k q^{n-k}}. \quad \square$

Corollary 3.10. $|\mathcal{C}_{a,*}^*| = q \sum_{1 \leq k \leq n} (-1)^{k-1} C(n, k)q^{(q-1)^k q^{n-k}}$

Proof. $\mathcal{C}_{a,*}^* = \bigcup_{b \in \mathbb{F}} \mathcal{C}_{a,b}^*$, we need to show $\mathcal{C}_{a,b_1}^* \cap \mathcal{C}_{a,b_2}^* = \emptyset$ if $b_1 \neq b_2$. Suppose $f \in \mathcal{C}_{a,b_1}^* \cap \mathcal{C}_{a,b_2}^*$, then there exist i_1 and $i_2 \in [n]$ such that $f \in \mathcal{C}_{a,b_1}^{i_1} \cap \mathcal{C}_{a,b_2}^{i_2}$ since $\mathcal{C}_{a,b}^* = \bigcup_{i \in [n]} \mathcal{C}_{a,b}^i$. If $i_1 = i_2$, we get a contradiction by Proposition 2.5. If $i_1 \neq i_2$, we get a contradiction by Proposition 2.6 since $\mathcal{C}_{a,b_1}^{i_1} \subset \mathcal{C}_{*,b_1}^{i_1}$ and $\mathcal{C}_{a,b_2}^{i_2} \subset \mathcal{C}_{*,b_2}^{i_2}$. \square

Now, we are going to find the formula for the number of all the canalyzing functions with given canalyzed value b . In other words, the formula of $|\mathcal{C}_{*,b}^*|$.

Let $S_b = \{\mathcal{C}_{a,b}^i | i \in [n], a \in \mathbb{F}\}$ for any $b \in \mathbb{F}$. By Inclusion and Exclusion Principle, we have $|\mathcal{C}_{*,b}^*| = |\bigcup_{i \in [n]} \mathcal{C}_{*,b}^i| = |\bigcup_{i \in [n]} \bigcup_{a \in \mathbb{F}} \mathcal{C}_{a,b}^i| = \sum_{k=1}^{nq} (-1)^{k-1} N_k$, where $N_k = \sum_{s \subset S_b, |s|=k} |\bigcap_{T \in s} T|$.

In order to evaluate N_k , we write all the elements in S_b as the following $n \times q$ matrix.

$$\begin{matrix} \mathcal{C}_{a_1,b}^1 & \mathcal{C}_{a_2,b}^1 & \dots & \mathcal{C}_{a_q,b}^1 \end{matrix}$$

$$\mathcal{C}_{a_1,b}^2 \quad \mathcal{C}_{a_2,b}^2 \cdots \mathcal{C}_{a_q,b}^2$$

$$\mathcal{C}_{a_1,b}^n \quad \mathcal{C}_{a_2,b}^n \cdots \mathcal{C}_{a_q,b}^n$$

For any $s \subset S_b$ with $|s| = k$, we will chose k elements from the above matrix to form s .

Suppose k_1 of its elements are from the first row (there are $C(q, k_1)$ many ways to do so). Let these k_1 elements be $\mathcal{C}_{a_{11},b}^1, \mathcal{C}_{a_{12},b}^1, \dots, \mathcal{C}_{a_{1k_1},b}^1$.

Suppose k_2 of its elements are from the second row (there are $C(q, k_2)$ many ways to do so). Let these k_2 elements be $\mathcal{C}_{a_{21},b}^2, \mathcal{C}_{a_{22},b}^2, \dots, \mathcal{C}_{a_{2k_2},b}^2$.

Suppose k_n of its elements are from the last row (there are $C(q, k_n)$ many ways to do so). Let these k_n elements be $\mathcal{C}_{a_{n1},b}^n, \mathcal{C}_{a_{n2},b}^n, \dots, \mathcal{C}_{a_{nk_n},b}^n$.

$$k_1 + k_2 + \dots + k_n = k, 0 \leq k_i \leq q, i = 1, 2, \dots, n.$$

Similarly to lemma 3.4, we have

Lemma 3.11. $f \in \bigcap_{T \in s} T$ iff $f = Q(\prod_{i=1}^{k_1}(x_1 - a_{1i}))(\prod_{i=1}^{k_2}(x_2 - a_{2i})) \dots (\prod_{i=1}^{k_n}(x_n - a_{ni})) + b$, where $\deg(Q)_i \leq q - k_i - 1, i = 1, 2, \dots, n$.

Similarly to Lemma 3.5, we have

Lemma 3.12. $|\bigcap_{T \in s} T| = q^{(q-k_1)(q-k_2) \dots (q-k_n)}$.

Hence,

$$N_k = \sum_{k_1+k_2+\dots+k_n=k} C(q, k_1)C(q, k_2) \dots C(q, k_n) q^{(q-k_1)(q-k_2) \dots (q-k_n)}.$$

We get

Theorem 3.13. For any $b \in F$, we have

$$|\mathcal{C}_{*,b}^*| = \sum_{k=1}^{nq} (-1)^{k-1} \left(\sum_{k_1+k_2+\dots+k_n=k} \left(\prod_{j=1}^n C(q, k_j) \right) q^{\prod_{j=1}^n (q-k_j)} \right)$$

In summary: For the following eight different classes of canalizing functions,

$\mathcal{C}_{a,b}^i, \mathcal{C}_{*,b}^i, \mathcal{C}_{a,*}^i, \mathcal{C}_{a,b}^*, \mathcal{C}_{*,*}^i, \mathcal{C}_{a,*}^*, \mathcal{C}_{*,b}^*, \mathcal{C}_{*,*}^*$ We have found the cardinality of all except for $\mathcal{C}_{*,*}^i$ and $\mathcal{C}_{*,*}^*$. We need more characterizations. We have

Lemma 3.14. Given $\{a_1, a_2, \dots, a_k\} \subset \mathbb{F}$ and $\{b_1, b_2, \dots, b_k\} \subset \mathbb{F}$, Then $f \in \bigcap_{j=1}^k \mathcal{C}_{a_j, b_j}^i$ iff $f = Q_k(X) \prod_{j=1}^k (x_i - a_j) + A_{k-1} \prod_{j=1}^{k-1} (x_i - a_j) + \dots + A_1(x_i - a_1) + A_0$. where $A_0 = b_1, a_t \in F, A_t$ is determined by $a_1, \dots, a_{t+1}, b_1, \dots, b_{t+1}, t = 1, 2, \dots, k-1$. $\deg(Q_k)_i \leq q - k - 1$.

Proof. For the necessity, we use induction principle.

For $k = 1$, it is true by the definition and Lemma 3.1

$f \in \bigcap_{j=1}^k \mathcal{C}_{a_j, b_j}^i$ implies $f \in \bigcap_{j=1}^{k-1} \mathcal{C}_{a_j, b_j}^i$, by the assumption, we have $f = Q_{k-1}(X) \prod_{j=1}^{k-1} (x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2} (x_i - a_j) + \dots + A_1(x_i - a_1) + A_0$.

Since $f \in \mathcal{C}_{a_k, b_k}^i$, we get

$f(x_1, \dots, x_{i-1}, a_k, x_{i+1}, \dots, x_n) = Q_{k-1} \prod_{j=1}^{k-1} (a_k - a_j) + A_{k-2} \prod_{j=1}^{k-2} (a_k - a_j) + \dots + A_1(a_k - a_1) + A_0 = b_k$ for any $X = (x_1, \dots, x_{i-1}, a_k, x_{i+1}, \dots, x_n)$. In other words,

$Q_{k-1}(x_1, \dots, x_{i-1}, a_k, x_{i+1}, \dots, x_n) = A_{k-1}$ for any $X = (x_1, \dots, x_{i-1}, a_k, x_{i+1}, \dots, x_n)$.

So $Q_{k-1} \in \mathcal{C}_{a_k, A_{k-1}}^i$. By Lemma 3.1, we have $Q_{k-1} = (x_i - a_k)Q_k + A_{k-1}$,

$$f = Q_{k-1}(X) \prod_{j=1}^{k-1} (x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2} (x_i - a_j) + \dots + A_1(x_i - a_1) + A_0 =$$

$$= ((x_i - a_k)Q_k + A_{k-1}) \prod_{j=1}^{k-1} (x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2} (x_i - a_j) + \dots + A_1(x_i - a_1) + A_0 =$$

$$= Q_k(X) \prod_{j=1}^k (x_i - a_j) + A_{k-1} \prod_{j=1}^{k-1} (x_i - a_j) + \dots + A_1(x_i - a_1) + A_0.$$

We finish the proof of necessity.

When $x_i = a_1$, we have $f = A_0 = b_1$, so $f \in \mathcal{C}_{a_1, b_1}^i$.

When $x_i = a_2$, we set $f = A_1(a_2 - a_1) + A_0 = b_2$, we get a unique solution for A_1 such that $f \in \mathcal{C}_{a_2, b_2}^i \dots$

When $x_i = a_k$, we get a unique solution for A_{k-1} such that $f \in \mathcal{C}_{a_k, b_k}^i$. In summary, $f \in \bigcap_{j=1}^k \mathcal{C}_{a_j, b_j}^i$. \square

From the above Lemma, we immediately obtain

Lemma 3.15. $|\bigcap_{j=1}^k \mathcal{C}_{a_j, b_j}^i| = q^{(q-k)q^{n-1}}$ given $\{a_1, a_2, \dots, a_k\} \subset \mathbb{F}$ and $\{b_1, b_2, \dots, b_k\} \subset \mathbb{F}$

In order to evaluate $|\mathcal{C}_{*,*}^i|$, we need to generalize Lemma 3.14.

To save space, we just focus on the cardinality in the following lemma.

Lemma 3.16. $a_{11}, a_{12}, \dots, a_{1k_1}; a_{21}, a_{22}, \dots, a_{2k_2}; \dots; a_{r1}, a_{r2}, \dots, a_{rk_r}$ are $k_1 + k_2 + \dots + k_r$ distinct elements of \mathbb{F} , $\{b_1, b_2, \dots, b_k\} \subset \mathbb{F}$. Then

$$|(\bigcap_{j=1}^{k_1} \mathcal{C}_{a_{1j}, b_1}^i) \cap (\bigcap_{j=1}^{k_2} \mathcal{C}_{a_{2j}, b_2}^i) \dots \cap (\bigcap_{j=1}^{k_r} \mathcal{C}_{a_{rj}, b_r}^i)| = q^{(q-k_1-k_2-\dots-k_r)q^{n-1}}$$

Proof. By Lemma 3.4, $f \in \bigcap_{j=1}^{k_1} \mathcal{C}_{a_{1j}, b_1}^i$ iff $f = Q(X) \prod_{j=1}^{k_1} (x_i - a_{1j}) + b_1$, $\deg(Q)_i \leq q - k_1 - 1$, i.e., we have

$$\bigcap_{j=1}^{k_1} \mathcal{C}_{a_{1j}, b_1}^i = \{Q(X) \prod_{j=1}^{k_1} (x_i - a_{1j}) + b_1 \mid \forall Q, \deg(Q)_i \leq q - k_1 - 1\}.$$

Let $f \in \bigcap_{j=1}^{k_1} \mathcal{C}_{a_{1j}, b_1}^i$, then $f = Q(X) \prod_{j=1}^{k_1} (x_i - a_{1j}) + b_1$. If we also have $f \in \mathcal{C}_{a_{21}, b_2}^i$, let $x_i = a_{21}$, we get

$f(x_1, \dots, x_{i-1}, a_{21}, x_{i+1}, \dots, x_n) = Q(x_1, \dots, x_{i-1}, a_{21}, x_{i+1}, \dots, x_n) \prod_{j=1}^{k_1} (a_{21} - a_{1j}) + b_1 = b_2$ for any $(x_1, \dots, x_{i-1}, a_{21}, x_{i+1}, \dots, x_n)$. The coefficient $\prod_{j=1}^{k_1} (a_{21} - a_{1j})$ is nonzero, so we can solve for Q and get $Q \in \mathcal{C}_{a_{21}, A_1}^i$ for some $A_1 \in \mathbb{F}$. Hence we can write

$Q = (x_i - a_{21})Q_1 + A_1$, i.e., $f = (x_i - a_{21})Q_1 \prod_{j=1}^{k_1} (x_i - a_{1j}) + O(x_i)$, where $O(x_i)$ is a one variable polynomial whose coefficients are completely determined by a_{ij}, b_i , $\deg(Q_1)_i = \deg(Q)_i - 1$, $\deg(O(x_i)) \leq k_1$.

Obviously, we can repeat the above process, to get

$$f \in (\bigcap_{j=1}^{k_1} \mathcal{C}_{a_{1j}, b_1}^i) \cap (\bigcap_{j=1}^{k_2} \mathcal{C}_{a_{2j}, b_2}^i) \dots \cap (\bigcap_{j=1}^{k_r} \mathcal{C}_{a_{rj}, b_r}^i) \text{ iff}$$

$f = \bar{Q}(\prod_{j=1}^{k_1} (x_i - a_{1j}))(\prod_{j=1}^{k_2} (x_i - a_{2j})) \dots (\prod_{j=1}^{k_r} (x_i - a_{rj})) + \bar{O}(x_i)$. Where $\deg(\bar{Q})_i \leq q - k_1 - k_2 - \dots - k_r - 1$ and $\bar{O}(x_i)$ is a uniquely determined polynomial of x_i and $\deg(\bar{O}(x_i)) \leq k_1 + \dots + k_r - 1$. Hence, we know the cardinality is $q^{(q-k_1-k_2-\dots-k_r)q^{n-1}}$. \square

Now, we are ready to find the cardinality of $\mathcal{C}_{*,*}^i$. We have

Theorem 3.17.

$$|\mathcal{C}_{*,*}^i| = q! \sum_{k=1}^q \frac{(-1)^{k-1} q^{(q-k)q^{n-1}}}{(q-k)!} \sum_{k_1 + \dots + k_q = k, 0 \leq k_i \leq q} \frac{1}{k_1! k_2! \dots k_q!}$$

Proof. $\mathcal{C}_{*,*}^i = \bigcup_{b \in \mathbb{F}} \mathcal{C}_{*,b}^i = \bigcup_{a \in \mathbb{F}} \bigcap_{b \in \mathbb{F}} \mathcal{C}_{a,b}^i$.

Let $S_i = \{\mathcal{C}_{a,b}^i \mid a, b \in \mathbb{F}\}$, we get $|\mathcal{C}_{*,*}^i| = \sum_{k=1}^{q^2} (-1)^{k-1} N_k$. Where $N_k = \sum_{s \subset S_i, |s|=k} |\bigcap_{T \in s} T|$. In order to evaluate N_k , we write all the elements in S_i as the following $q \times q$ matrix.

$$\begin{matrix} \mathcal{C}_{a_1, b_1}^i & \mathcal{C}_{a_2, b_1}^i & \dots & \mathcal{C}_{a_q, b_1}^i \\ \mathcal{C}_{a_1, b_2}^i & \mathcal{C}_{a_2, b_2}^i & \dots & \mathcal{C}_{a_q, b_2}^i \\ \dots & \dots & \dots & \dots \end{matrix}$$

$$\mathcal{C}_{a_1, b_q}^i \mathcal{C}_{a_2, b_q}^i \dots \mathcal{C}_{a_q, b_q}^i$$

For any $s \subset S_i$ with $|s| = k$, we will chose k elements from the above matrix to form s .

Suppose k_1 of its elements are from the first row (there are $C(q, k_1)$ many ways to do so). Let these k_1 elements be $\mathcal{C}_{a_{11}, b_1}^i, \mathcal{C}_{a_{12}, b_1}^i, \dots, \mathcal{C}_{a_{1k_1}, b_1}^i$.

Suppose k_2 of its elements are from the second row, we must chose these elements from different columns, otherwise the intersection will be ϕ by Proposition 2.5 (there are $C(q - k_1, k_2)$ many ways to do so). Let these k_2 elements be $\mathcal{C}_{a_{21}, b_2}^i, \mathcal{C}_{a_{22}, b_2}^i, \dots, \mathcal{C}_{a_{2k_2}, b_2}^i$

.....
Suppose k_q of its elements are from the last row (there are $C(q - k_1 - k_2 - \dots - k_{q-1}, k_q)$ many ways to do so). Let these k_q elements be $\mathcal{C}_{a_{q1}, b_q}^i, \mathcal{C}_{a_{q2}, b_q}^i, \dots, \mathcal{C}_{a_{qk_q}, b_q}^i$.

$$k_1 + k_2 + \dots + k_q = k, 0 \leq k_i \leq q, i = 1, 2, \dots, q.$$

$$N_k = \sum_{s \subset S_i, |s|=k} |\bigcap_{T \in s} T|$$

$$= \sum_{k_1 + \dots + k_q = k, 0 \leq k_i \leq q} C(q, k_1) C(q - k_1, k_2) \dots C(q - k_1 - \dots - k_{q-1}, k_q) I_{k_1 k_2 \dots k_q}, \text{ where}$$

$$I_{k_1 k_2 \dots k_q} = |(\bigcap_{j=1}^{k_1} \mathcal{C}_{a_{1j}, b_1}^i) \cap (\bigcap_{j=1}^{k_2} \mathcal{C}_{a_{2j}, b_2}^i) \dots \cap (\bigcap_{j=1}^{k_q} \mathcal{C}_{a_{qj}, b_q}^i)|.$$

By Lemma 3.16, we know $I_{k_1 k_2 \dots k_q} = q^{(q-k_1-k_2-\dots-k_q)q^{n-1}} = q^{(q-k)q^{n-1}}$, this number is zero if $k > q$.

A straightforward computation shows that

$$C(q, k_1) C(q - k_1, k_2) \dots C(q - k_1 - \dots - k_{q-1}, k_q) = \frac{q!}{k_1! k_2! \dots k_q! (q-k)!}.$$

$$\text{Hence, we get } |\mathcal{C}_{*,*}^i| = \sum_{k=1}^{q^2} (-1)^{k-1} N_k = \sum_{k=1}^q (-1)^{k-1} N_k =$$

$$\begin{aligned} &= \sum_{k=1}^q (-1)^{k-1} \sum_{k_1 + \dots + k_q = k, 0 \leq k_i \leq q} \frac{q!}{k_1! k_2! \dots k_q! (q-k)!} q^{(q-k)q^{n-1}} = \\ &= q! \sum_{k=1}^q \frac{(-1)^{k-1} q^{(q-k)q^{n-1}}}{(q-k)!} \sum_{k_1 + \dots + k_q = k, 0 \leq k_i \leq q} \frac{1}{k_1! k_2! \dots k_q!} \end{aligned}$$

□

Now we begin to evaluate $|\mathcal{C}_{*,*}^*|$. We have

Theorem 3.18.

$$|\mathcal{C}_{*,*}^*| = \sum_{k=1}^q (-1)^{k-1} U_k + \sum_{k=1}^{nq} (-1)^{k-1} V_k.$$

where

$$\begin{aligned} U_k &= n \sum_{t_1 + t_2 + \dots + t_q = k, 0 \leq t_i \leq q} \frac{q!}{t_1! t_2! \dots t_q! (q-k)!} q^{(q-k)q^{n-1}} = \\ &= \frac{nq!}{(q-k)!} q^{(q-k)q^{n-1}} \sum_{t_1 + t_2 + \dots + t_q = k, 0 \leq t_i \leq q} \frac{1}{t_1! t_2! \dots t_q!} \end{aligned}$$

$$V_k = q \sum_{k_1 + \dots + k_n = k, 0 \leq k_i \leq k-1, 0 \leq k_i \leq q} \left(\prod_{j=1}^n C(q, k_j) \right) q^{\prod_{j=1}^n (q-k_j)}$$

Proof. First $\mathcal{C}_{*,*}^* = \bigcup_{i \in [n]} \bigcup_{a \in \mathbb{F}} \bigcup_{b \in \mathbb{F}} \mathcal{C}_{a,b}^i$, Let $S = \{\mathcal{C}_{a,b}^i | i \in [n], a, b \in F\}$,

then $|\mathcal{C}_{*,*}^*| = \sum_{k=1}^{nq^2} (-1)^{k-1} N_k$, where $N_k = \sum_{s \subset S, |s|=k} |\bigcap_{T \in s} T|$.

We write all the nq^2 elements of S as the following n many $q \times q$ matrices.

$$\begin{aligned} &\mathcal{C}_{a_1, b_1}^1 \mathcal{C}_{a_1, b_2}^1 \dots \mathcal{C}_{a_1, b_q}^1 \\ &\mathcal{C}_{a_2, b_1}^1 \mathcal{C}_{a_2, b_2}^1 \dots \mathcal{C}_{a_2, b_q}^1 \end{aligned}$$

$$\dots\dots\dots$$

$$\mathcal{C}_{a_q,b_1}^1 \mathcal{C}_{a_q,b_2}^1 \dots \mathcal{C}_{a_q,b_q}^1$$

We call this matrix M_1 .

$$\mathcal{C}_{a_1,b_1}^2 \mathcal{C}_{a_1,b_2}^2 \dots \mathcal{C}_{a_1,b_q}^2$$

$$\mathcal{C}_{a_2,b_1}^2 \mathcal{C}_{a_2,b_2}^2 \dots \mathcal{C}_{a_2,b_q}^2$$

$$\dots\dots\dots$$

$$\mathcal{C}_{a_q,b_1}^2 \mathcal{C}_{a_q,b_2}^2 \dots \mathcal{C}_{a_q,b_q}^2$$

We call this matrix M_2 .

$$\dots\dots\dots$$

$$\dots\dots\dots$$

$$\dots\dots\dots$$

$$\mathcal{C}_{a_1,b_1}^n \mathcal{C}_{a_1,b_2}^n \dots \mathcal{C}_{a_1,b_q}^n$$

$$\mathcal{C}_{a_2,b_1}^n \mathcal{C}_{a_2,b_2}^n \dots \mathcal{C}_{a_2,b_q}^n$$

$$\dots\dots\dots$$

$$\mathcal{C}_{a_q,b_1}^n \mathcal{C}_{a_q,b_2}^n \dots \mathcal{C}_{a_q,b_q}^n$$

We call this matrix M_n .

We combine all the above $M_i, i = 1, 2, \dots, n$. to form a $nq \times q$ matrix M whose first q rows are M_1 , the second q rows are M_2, \dots , the last q rows are M_n .

We are going to chose k elements from M to form the intersection. In order to get a possible non empty intersection, we know all these k elements must come from either the same M_i (for some fixed i) or all of them from the same column of M by Proposition 2.6. Inside the fixed M_i , each elements must come from different rows by Proposition 2.5.

Hence, a typical intersection either looks like the one in Lemma 3.16 or the one in Lemma 3.11. But these two cases are not disjoint.

Suppose we chose k_i elements from $M_i, i = 1, 2, \dots, n, k_1 + k_2 + \dots + k_n = k, 0 \leq k_i \leq k, i = 1, 2, \dots, n$.

If there exist i such that $k_i = k$, then $k_j = 0, \forall j \neq i$. This implies the intersection looks like the one in Lemma 3.16 and $k \leq q$.

If $0 \leq k_i \leq k-1, \forall i \in [n]$, then the intersection looks like the one in Lemma 3.11 and $k \leq nq$.

The above two cases are disjoint now. By Lemma 3.16 and Lemma 3.12, we get

$$N_k = \sum_{s \subset S, |s|=k} |\bigcap_{T \in s} T| = \sum_{k_1 + \dots + k_n = k, 0 \leq k_i \leq k} = \sum_{\exists i, k_i = k} + \sum_{k_i \leq k-1, i=1, \dots, n} = U_k + V_k$$

where

$$U_k = n \sum_{t_1 + t_2 + \dots + t_q = k, 0 \leq t_i \leq q} \frac{q!}{t_1! t_2! \dots t_q! (q-k)!} q^{(q-k)q^{n-1}} =$$

$$\frac{nq!}{(q-k)!} q^{(q-k)q^{n-1}} \sum_{t_1 + t_2 + \dots + t_q = k, 0 \leq t_i \leq q} \frac{1}{t_1! t_2! \dots t_q!}$$

$$V_k = q \sum_{k_1 + \dots + k_n = k, 0 \leq k_i \leq k-1, 0 \leq k_i \leq q} \left(\prod_{j=1}^n C(q, k_j) \right) q^{\prod_{j=1}^n (q-k_j)}$$

Hence,

$$|\mathcal{C}_{*,*}^*| = \sum_{k=1}^{nq^2} (-1)^{k-1} N_k = \sum_{k=1}^{nq^2} (-1)^{k-1} (U_k + V_k) = \sum_{k=1}^q (-1)^{k-1} U_k + \sum_{k=1}^{nq} (-1)^{k-1} V_k,$$

□

In the following, we will reduce the formula $|\mathcal{C}_{*,*}^*|$ when $q = 2$ and compare it with the one in [2].

$$|\mathcal{C}_{*,*}^*| = \sum_{k=1}^2 (-1)^{k-1} U_k + \sum_{k=1}^{2n} (-1)^{k-1} V_k.$$

where

$$U_k = n \sum_{t_1+t_2=k, 0 \leq t_i \leq 2} \frac{2!}{t_1! t_2! (2-k)!} 2^{(2-k)2^{n-1}}$$

$$V_k = 2 \sum_{k_1+\dots+k_n=k, 0 \leq k_i \leq k-1, 0 \leq k_i \leq 2} \left(\prod_{j=1}^n C(2, k_j) \right) 2^{\prod_{j=1}^n (2-k_j)}$$

A simple calculation shows that $U_1 = 4n2^{2^{n-1}} = C(n, 1)2^2 2^{2^{n-1}}$ and $U_2 = 4n$.
 $V_1 = 0$ since the condition of the sum is not satisfied.

$$V_2 = 2 \sum_{k_1+\dots+k_n=2, 0 \leq k_i \leq 1} \left(\prod_{j=1}^n C(2, k_j) \right) 2^{\prod_{j=1}^n (2-k_j)} = C(n, 2)2^3 2^{2^{n-2}}$$

When $3 \leq k \leq 2n$

$$V_k = 2 \sum_{k_1+\dots+k_n=k, 0 \leq k_i \leq 2} \left(\prod_{j=1}^n C(2, k_j) \right) 2^{\prod_{j=1}^n (2-k_j)}$$

$$= C(n, k)2^{k+1}2^{2^{n-k}} + \sum_{1 \leq t \leq \lfloor \frac{k}{2} \rfloor} C(n, t)C(n-t, k-2t)2^{k-2t+1}$$

Hence, when $q = 2$,

$$|\mathcal{C}_{*,*}^*| = -4n + \sum_{1 \leq k \leq n} (-1)^{k+1} C(n, k)2^{k+1}2^{2^{n-k}}$$

$$+ \sum_{3 \leq k \leq 2n} (-1)^{k-1} \sum_{1 \leq t \leq \lfloor \frac{k}{2} \rfloor} C(n, t)C(n-t, k-2t)2^{k-2t+1}$$

When $n=1, 2, 3, 4$, one can obtain (without calculator) the sequence 4, 14, 120, 3514. These results are consistent with those in [2]. By [2], the cardinality of $\mathcal{C}_{*,*}^*$ should be

$$|\mathcal{C}_{*,*}^*| = 2((-1)^n - n) + \sum_{1 \leq k \leq n} (-1)^{k+1} C(n, k)2^{k+1}2^{2^{n-k}}.$$

So, we obtain the following combinatorial identity (for any positive integer n).

$$\sum_{3 \leq k \leq 2n} (-1)^{k-1} \sum_{1 \leq t \leq \lfloor \frac{k}{2} \rfloor} C(n, t)C(n-t, k-2t)2^{k-2t+1} = 2((-1)^n + n)$$

The left sum should be explained as 0 if $n = 1$. As usual, $C(n, k)$ is 0 if $k > n$.

For general q , from Lemma 3.2, we know $|\mathcal{C}_{a,b}^i| = q^{q^n - q^{n-1}}$, since $\mathcal{C}_{*,*}^* = \bigcup_{i \in [n]} \bigcup_{a \in \mathbb{F}} \bigcup_{b \in \mathbb{F}} \mathcal{C}_{a,b}^i$, we obtain $|\mathcal{C}_{*,*}^*| \leq nq^2 q^{(q-1)q^{n-1}}$.

In order to get an intuitive idea about the magnitude of all the cardinality numbers, We will find their asymptote as $n \rightarrow \infty$ or $q \rightarrow \infty$.

We have the following notation

Definition 3.19. $f(x) \stackrel{x}{\cong} g(x)$ if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$.

Now, we can list all the cardinalities asymptotically, we have

Theorem 3.20.

$$\begin{aligned}
|\mathcal{C}_{a,b}^i| &= q^{(q-1)q^{n-1}}; \\
|\mathcal{C}_{a,*}^i| &= qq^{(q-1)q^{n-1}}; \\
|\mathcal{C}_{*,b}^i| &\stackrel{n}{\cong} qq^{(q-1)q^{n-1}}, |\mathcal{C}_{*,b}^i| \stackrel{q}{\cong} qq^{(q-1)q^{n-1}}; \\
|\mathcal{C}_{a,b}^*| &\stackrel{n}{\cong} nq^{(q-1)q^{n-1}}, |\mathcal{C}_{a,b}^*| \stackrel{q}{\cong} nq^{(q-1)q^{n-1}}; \\
|\mathcal{C}_{a,*}^*| &\stackrel{n}{\cong} nqq^{(q-1)q^{n-1}}, |\mathcal{C}_{a,*}^*| \stackrel{q}{\cong} nqq^{(q-1)q^{n-1}}; \\
|\mathcal{C}_{*,b}^*| &\stackrel{n}{\cong} nqq^{(q-1)q^{n-1}}, |\mathcal{C}_{*,b}^*| \stackrel{q}{\cong} nqq^{(q-1)q^{n-1}}; \\
|\mathcal{C}_{*,*}^i| &\stackrel{n}{\cong} q^2 q^{(q-1)q^{n-1}}, |\mathcal{C}_{*,*}^i| \stackrel{q}{\cong} q^2 q^{(q-1)q^{n-1}}; \\
|\mathcal{C}_{*,*}^*| &\stackrel{n}{\cong} nq^2 q^{(q-1)q^{n-1}}, |\mathcal{C}_{*,*}^*| \stackrel{q}{\cong} nq^2 q^{(q-1)q^{n-1}};
\end{aligned}$$

Proof. The first two rows are previous lemma and corollary.

We will give a proof for the last row, the others are similar and easier.

$$\begin{aligned}
|\mathcal{C}_{*,*}^*| &= \sum_{k=1}^q (-1)^{k-1} U_k + \sum_{k=1}^{nq} (-1)^{k-1} V_k. \\
U_1 &= \frac{nq!}{(q-1)!} q^{(q-1)q^{n-1}} \sum_{t_1+t_2+\dots+t_q=1, 0 \leq t_i \leq q} \frac{1}{t_1! t_2! \dots t_q!} = nq^2 q^{(q-1)q^{n-1}}.
\end{aligned}$$

When $2 \leq k \leq q$, we have

$$\begin{aligned}
U_k &= \frac{nq!}{(q-k)!} q^{(q-k)q^{n-1}} \sum_{t_1+t_2+\dots+t_q=k, 0 \leq t_i \leq q} \frac{1}{t_1! t_2! \dots t_q!} \leq nq! q^{(q-2)q^{n-1}} \sum_{0 \leq t_i \leq q, i=1,2,\dots,q} 1 \\
&= nq! q^{(q-2)q^{n-1}} (q+1)^q.
\end{aligned}$$

So, $\lim_{n \rightarrow \infty} \frac{U_k}{U_1} = 0$ for $2 \leq k \leq q$.

$V_1 = 0$ since the condition of the sum is not satisfied.

When $2 \leq k \leq nq$, we have

$$\begin{aligned}
V_k &= q \sum_{k_1+\dots+k_n=k, 0 \leq k_i \leq k-1, 0 \leq k_i \leq q} \left(\prod_{j=1}^n C(q, k_j) \right) q^{\prod_{j=1}^n (q-k_j)} \leq q \sum_{0 \leq k_i \leq q, i=1,2,\dots,n} (nq!) q^{(q-1)^2 q^{n-2}} \\
&= q(q+1)^n nq! q^{(q-1)^2 q^{n-2}}.
\end{aligned}$$

Hence,

$$\left| \sum_{k=1}^{nq} (-1)^{k-1} V_k \right| \leq nqq(q+1)^n nq! q^{(q-1)^2 q^{n-2}}.$$

We obtain

$$\lim_{n \rightarrow \infty} \frac{|\sum_{k=1}^{nq} (-1)^{k-1} V_k|}{U_1} \leq \lim_{n \rightarrow \infty} \frac{nqq(q+1)^n nq! q^{(q-1)^2 q^{n-2}}}{nq^2 q^{(q-1)q^{n-1}}} = \lim_{n \rightarrow \infty} \frac{(q+1)^n nq!}{q^{(q-1)q^{n-2}}} = 0.$$

In summary, we obtain

$$\lim_{n \rightarrow \infty} \frac{|\mathcal{C}_{*,*}^*|}{nq^2 q^{(q-1)q^{n-1}}} = 1$$

In other words,

$$|\mathcal{C}_{*,*}^*| \stackrel{n}{\cong} nq^2 q^{(q-1)q^{n-1}}.$$

From the above proof, it is also clear that we have

$$\lim_{q \rightarrow \infty} \frac{|\mathcal{C}_{*,*}^*|}{nq^2 q^{(q-1)q^{n-1}}} = 1$$

In other words,

$$|\mathcal{C}_{*,*}^*| \stackrel{q}{\cong} nq^2 q^{(q-1)q^{n-1}}.$$

□

When $q = 2$, the first part of the last row in the above theorem has been obtained in [2].

4. CONCLUSION

In this paper, we generalized the definition of Boolean canalizing functions to the functions over general finite fields \mathbb{F}_q . We obtain clear characterization for all eight classes of canalizing functions. Using Inclusion and Exclusion Principle, we also obtain eight formulas for the cardinality of these classes. The main idea is from [1] and [2]. Actually, the characterization is motivated from a simple lemma in [1]. The enumeration idea is a natural extension of [2]. By specifying our results to the case $q = 2$, we obtain the formula in [2], and derive an interesting combinatorial identity. Finally, for a better understanding to the magnitudes, we provide all the eight asymptotes of these cardinalities as either $n \rightarrow \infty$ or $q \rightarrow \infty$.

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